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Generalized Modus Ponens using Fodor's Implication and T-norm Product with Threshold

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Abstract: Using Generalized Modus Ponens reasoning, we examine the values of the inferred conclusion depending on the correspondence between the premise of the rule and the observed fact. The conclusion is obtained using Fodor's implication in order to represent a fuzzy if-then rule with a single input single output and the t-norm with threshold generated by t-norm product, as a compositional operator. A comparison study with the case when the standard t-norm product is used is made. Some comments and an example are presented in order to show how the obtained results can be used.
Keywords: t-norm, t-conorm, negation, implication, fuzzy number, generalized modus ponens rule

1 Introduction

The database of a rule-based system may contain imprecisions which appear in the description of the rules given by the expert. The imprecision implies the difficulty of representing the rules expressed, generally, by means of natural language. Another difficulty is the utilization of these rules in approximate reasoning when the observed facts do not match the condition of the rule. In order to obtain an imprecise conclusion from imprecise premises, Zadeh extends the traditional Modus Ponens rule obtaining Generalized Modus Ponens (GMP). An investigation of GMP inference was made by many papers: [2], [3], [4], [5], [7], [9], [14], [15], [27], [28], [29], [30], [33], [34], [35]. Also, we analyzed this type of inference in some papers: [19] , [22], [24] , [25], [26].
The proposition

$$X\ is\ A$$

can be understood as

$$the\ quantity\ X\ satisfies\ the\ predicate\ A$$

or

$$the\ variable\ X\ takes\ its\ values\ in\ the\ set\ A.$$

The semantic content of the proposition

$$X\ is\ A$$

can be represented by

$$\pi_X = \mu_A,$$

where π_X is the possibility distribution restricting the possible value of X and μ_A is the membership function of the set A.

Because the majority of practical applications work with trapezoidal or triangular distributions and these representations are still a subject of various recent papers ([1], [13] and [16], for instance) we will work with membership functions represented by trapezoidal fuzzy numbers. Such a number $N = (a, b, \alpha, \beta)$ is defined as

$$\mu_N(x) = \begin{cases} 0 \;\; for \;\; x < a - \alpha \\ \dfrac{x - a + \alpha}{\alpha} \;\; for \;\; x \in [a - \alpha, a] \\ 1 \;\; for \;\; x \in [a, b] \\ \dfrac{b + \beta - x}{\beta} \;\; for \;\; x \in [b, b + \beta] \\ 0 \;\; for \;\; x > b + \beta \end{cases}$$

Let X and Y be two variables whose domains are U and V, respectively. A causal link from X to Y is represented as a conditional possibility distribution [35, 36] $\pi_{Y/X}$ which restricts the possible values of Y for a given value of X. For the rule

$$if \; X \; is \; A \; then \; Y \; is \; B$$

we have

$$\forall \; u \; \in U, \; \forall \; v \; \in V, \; \pi_{Y/X}(v, u) = \mu_A(u) \to \mu_B(v)$$

where $\to$ is an implication operator and μ_A and μ_B are the possibility distributions of the propositions "$X \; is \; A$" and "$Y \; is \; B$", respectively.
If $\mu_{A'}$ is the possibility distribution of the proposition

$$X \; is \; A'$$

then from the rule

$$if \; X \; is \; A \; then \; Y \; is \; B$$

and the fact

$$X \; is \; A'$$

Generalized Modus Ponens rule computes the possibility distribution $\mu_{B'}$ of the conclusion

$$Y \; is \; B'$$

as

$$\mu_{B'}(v) = \sup_{u \in U} T\left(\mu_{A'}(u), \pi_{Y/X}(v, u)\right),$$

where T is a t-norm.

2 Basic concepts

The main concepts used in GMP are presented below, using the terminology of [8], [17] and [32].

Definition 1. A function $T : [0,1]^2 \to [0,1]$ is a t-norm iff it is commutative, associative, non-decreasing and $T(x, 1) = x \; \forall x \in [0,1]$.

Definition 2. A function $S : [0,1]^2 \to [0,1]$ is a t-conorm iff it is commutative, associative, non-decreasing and $S(x, 0) = x \; \forall x \in [0,1]$.

Definition 3. A function $N : [0,1] \to [0,1]$ is a strong negation iff it is an involutive and continuous decreasing function from $[0,1]$ to itself.

In order to represent a rule, the notion of fuzzy implication is used. We recall an axiomatic approach (formulated by Fodor in [10, 11, 12]) to the definition of fuzzy implication.

Definition 4. An implication is a function $I : [0,1]^2 \rightarrow [0,1]$ satisfying the following conditions:

I1: If $x \leq z$ then $I(x,y) \geq I(z,y)$ for all $x,y,z \in [0,1]$

I2: If $y \leq z$ then $I(x,y) \leq I(x,z)$ for all $x,y,z \in [0,1]$

I3: $I(0,y) = 1$ (falsity implies anything) for all $y \in [0,1]$

I4: $I(x,1) = 1$ (anything implies tautology) for all $x \in [0,1]$

I5: $I(1,0) = 0$ (Booleanity)

The following properties could be important in some applications:

I6: $I(1,x) = x$ (tautology cannot justify anything) for all $x \in [0,1]$

I7: $I(x,I(y,z)) = I(y,I(x,z))$ (exchange principle) for all $x,y,z \in [0,1]$

I8: $x \leq y$ if and only if $I(x,y) = 1$ (implication defines ordering) for all $x,y \in [0,1]$

I9: $I(x,0) = N(x)$ for all $x \in [0,1]$ is a strong negation

I10: $I(x,y) \geq y$ for all $x,y \in [0,1]$

I11: $I(x,x) = 1$ (identity principle) for all $x \in [0,1]$

I12: $I(x,y) = I(N(y),N(x))$ for all $x,y \in [0,1]$ and a strong negation N

I13: I is a continuous function.

The most important families of implications are given by

Definition 5. A $\mathtt{S}$-implication associated with a t-conorm S and a strong negation N is defined by

$$I_{\mathtt{S}}^{S,N}(x,y) = S(N(x),y)\ \forall x,y \in [0,1]$$

A $\mathtt{R}$-implication associated with a t-norm T is defined by

$$I_{\mathtt{R}}^{T}(x,y) = \sup\{z \in [0,1] | T(x,z) \leq y\}\ \forall x,y \in [0,1]$$

A $\mathtt{QL}$-implication is defined by

$$I_{\mathtt{QL}}^{T,S,N}(x,y) = S(N(x),T(x,y))\ \forall x,y \in [0,1]$$

One of the most important implications is the Fodor's implication

$$I_F(x,y) = \begin{cases} 1 & if\ \ x \leq y \\ max(1-x,y) & otherwise \end{cases}$$

which is [5] a $\mathtt{R}$-implication for $T = min_0$, a $\mathtt{S}$ -implication for $S = max_0$ and a $\mathtt{QL}$-implication for $T = min$ and $S = max_0$, where

$$min_0(x,y) = \begin{cases} 0 & if\ \ x+y \leq 1 \\ min(x,y) & if\ \ x+y > 1 \end{cases}$$

and

$$max_0(x,y) = \begin{cases} 1 & if\ \ x+y \geq 1 \\ max(x,y) & if\ \ x+y < 1 \end{cases}$$

and $N(x) = 1 - x$. Besides, the Fodor's implication verifies the properties I1-I12. An important class of t-norms (t-conorms) is given by the t-norms (t-conorms) with thresholds, obtained from standard t-norms (t-conorms); the number of thresholds is an integer $n \geq 1$. First example of operators with 1-threshold were given by Pacholczyk in [31]. Various families of such t-operators can be found in [18, 20, 21, 23], where the advantage of their usage to represent the uncertain knowledge is justified. In this paper we analyze the results obtained by reasoning with imprecise knowledge using a t-norm with threshold as a composition operator. Finally we will compare these results with those obtained using the corresponding standard operators. We consider the following t-norm with a single threshold $k \in (0,1)$[31]

$$T_k(x,y) = \begin{cases} \frac{k}{1-k}T(\frac{1-k}{k}x, \frac{1-k}{k}y) & if \;\; x \leq k \;\; and \;\; y \leq k \\ min(x,y) & if \;\; x > k \;\; or \;\; y > k \end{cases}$$

obtained from the t-norm $T(x,y)$. We will work with the t-norm generated by $T_P(x,y) = xy$, which is one of the most used; it results

$$T_k(x,y) = \begin{cases} \frac{1-k}{k}xy & if \;\; x \leq k \;\; and \;\; y \leq k \\ min(x,y) & if \;\; x > k \;\; or \;\; y > k \end{cases}$$

3 Main results

Taking into account the following reasons, we shall work with rules having a single input single output:

a) a rule with multiple consequent can be treated as a set of rules with a single conclusion; for instance, the rule

$$if \; antecedent \; then \; C_1 \; and \; C_2 \; and \; .... \; and \; C_n$$

is equivalent to the rules

$$if \; antecedent \; then \; C_1$$

$$if \; antecedent \; then \; C_2$$

....

$$if \; antecedent \; then \; C_n.$$

b) a rule with multiple premise can be broken up into simple rules [6] when the rules are represented with any S-implication or any R-implication and the observations are normalized fuzzy sets.
Our aim is to obtain the conclusion "$Y \; is \; B'$" from the rule

$$if \; X \; is \; A \; then \; Y \; is \; B$$

and the fact

$$X \; is \; A'$$

where the fuzzy sets A, A', B and B' are represented by trapezoidal possibility distributions. The set B' is computed as

$$\mu_{B'}(v) = \sup_{u \in U} T_k(\mu_{A'}(u), I_F(\mu_A(u), \mu_B(v))),$$

analyzing five cases, depending on the relation between μ_A and $\mu_{A'}$.

Theorem 6. *If the premise contains the observation, i. e. $\mu_{A'}(u) \le \mu_A(u)\ \forall u \in U$, then*

$$\mu_{B'}(v) = \mu_B(v)\ if\ \mu_B(v) \ge 0.5$$

$$\mu_{B'}(v) \in [\mu_B(v), 1-\mu_B(v))\ if\ \mu_B(v) < 0.5$$

Proof. i1) value on the set $U_1 = \{u \in U/\mu_A(u) \le \mu_B(v)\}$
Because $I_F(\mu_A(u), \mu_B(v)) = 1$, we have

$$\mu_{B'}(v) = \sup_{u\in U_1} T_k(\mu_{A'}(u), 1) = \sup_{u\in U_1} \mu_{A'}(u) \le \mu_B(v).$$

i2) value on the set

$U_2 = \{u \in U/\mu_A(u) > \mu_B(v) \ge 0.5\} \cup \{u \in U/\mu_A(u) > 1-\mu_B(v) > 0.5\}$

We have $I_F(\mu_A(u), \mu_B(v)) = \mu_B(v)$. If $k < \mu_B(v)$ then

$$\mu_{B'}(v) = \sup_{u\in U_2} T_k(\mu_{A'}(u), \mu_B(v)) = \sup_{u\in U_2} min(\mu_{A'}(u), \mu_B(v)) = \mu_B(v).$$

For $k \ge \mu_B(v)$ and $U_2^1 = \{u \in U_2/\mu_{A'}(u) \le k\}$ we have

$$\mu_{B'}(v) = \sup_{u\in U_2^1} T_k(\mu_{A'}(u), \mu_B(v)) = \sup_{u\in U_2^1} \frac{1-k}{k}\mu_{A'}(u)\mu_B(v) \le (1-k)\mu_B(v) < \mu_B(v).$$

For $k \ge \mu_B(v)$ and $U_2^2 = \{u \in U_2/\mu_{A'}(u) > k\}$ we obtain

$$\mu_{B'}(v) = \sup_{u\in U_2^2} T_k(\mu_{A'}(u), \mu_B(v)) = \sup_{u\in U_2^2} min(\mu_{A'}(u), \mu_B(v)) = \mu_B(v).$$

i3) value on the set $U_3 = \{u \in U/\mu_B(v) < \mu_A(u) \le 1-\mu_B(v)\}$
In this case $I_F(\mu_A(u), \mu_B(v)) = 1-\mu_A(u)$ and therefore

$$\mu_{B'}(v) = \sup_{u\in U_3} T_k(\mu_{A'}(u), 1-\mu_A(u)).$$

For $k < \mu_B(v)$ we have $1-\mu_A(u) \ge \mu_B(v) > k$ and $T_k \equiv min$. It results

$$\mu_{B'}(v) = \sup_{u\in U_3} min(\mu_{A'}(u), 1-\mu_A(u)) < 1-\mu_B(v).$$

For $\mu_B(v) \le k \le 1-\mu_B(v)$ we analyze the cases:
$i3_1$): value on the set $U_3^1 = \{u \in U/\mu_B(v) \le \mu_A(u) < 1-k\}$
Because $k < 1-\mu_A(u)$ we obtain

$$\begin{aligned}\mu_{B'}(v) &= \sup\nolimits_{u\in U_3^1} T_k(\mu_{A'}(u), 1-\mu_A(u)) \\ &= \sup\nolimits_{u\in U_3^1} min(\mu_{A'}(u), 1-\mu_A(u)) < min(1-k, 1-\mu_B(v)) = 1-k.\end{aligned}$$

$i3_2$): value on the set $U_3^2 = \{u \in U/\mu_B(v) < 1-k \le \mu_A(u) \le 1-\mu_B(v)\}$
In this case, $1-\mu_A(u) \le k$ and we study three possibilities, depending on $\mu_{A'}(u)$.
$i3_2^1$): on the set $U_3^{2,1} = \{u \in U_3^2/\mu_{A'}(u) = 0\}$ we obtain $\mu_{B'}(v) = 0$
$i3_2^2$): on the set $U_3^{2,2} = \{u \in U_3^2/\mu_{A'}(u) \in (0,k]\}$ we have

$$\mu_{B'}(v) = \sup_{u\in U_3^{2,2}} \frac{1-k}{k}\mu_{A'}(u)(1-\mu_A(u)) < k(1-k).$$

$i3_2^3$): on the set $U_3^{2;3} = \{u \in U_3^2 / \mu_{A'}(u) > k\}$ we get

$$\mu_{B'}(v) = \sup_{u \in U_3^{2;3}} min(\mu_{A'}(u), 1 - \mu_A(u)) < 1 - \mu_B(v).$$

For $k > 1 - \mu_B(v)$ we consider the set

$$U_3^3 = \{u \in U_3 / \mu_B(v) > 1 - k\} = \{u \in U / 1 - k < \mu_B(v) < \mu_A(u) \le 1 - \mu_B(v)\}$$

and we work with the subsets of U_3^3 for which $\mu_{A'}(u) = 0$, $\mu_{A'}(u) \in (0, k]$ and $\mu_{A'}(u) > k$, respectively; we obtain the following corresponding results:

$$\mu_{B'}(v) = 0,\ \mu_{B'}(v) < k(1-k)\ and\ \mu_{B'}(v) < 1 - \mu_B(v).$$

Synthesizing the previous results, one obtain the conclusion formulated in the theorem. □

Theorem 7. *If the premise and the observation coincide, i. e.* $\mu_A(u) = \mu_{A'}(u)\ \forall u \in U$, *then*

$$\mu_{B'}(v) = \mu_B(v)\ if\ k > 0.5\ and\ \mu_B(v) \ge 1 - k,$$

$$\mu_{B'}(v) \in [\mu_B(v), 1 - k)\ if\ k > 0.5\ and\ \mu_B(v) < 1 - k,$$

$$\mu_{B'}(v) = max(0.5, \mu_B(v))\ if\ k \le 0.5.$$

Proof. In this case one repeat the proof of the Theorem 6 taking account the equality $\mu_A(u) = \mu_{A'}(u)$ $\forall u \in U$. It results:

1) if $0.5 < k \le \mu_B(v)$ then $\mu_{B'}(v) = \mu_B(v)$
2) if $k \le 0.5 \le \mu_B(v)$ then $\mu_{B'}(v) = \mu_B(v)$
3) if $k \le \mu_B(v) < 0.5$ then $\mu_{B'}(v) = 0.5$
4) if $\mu_B(v) \le k \le 0.5$ then $\mu_{B'}(v) = 0.5$
5) if $0.5 \le \mu_B(v) < k$ then $\mu_{B'}(v) = \mu_B(v)$
6) if $\mu_B(v) \le 0.5 < k$ then $\mu_{B'}(v) = \mu_B(v)\ if\ \mu_B(v) \ge 1 - k$ and

$$\mu_{B'}(v) \in [\mu_B(v), 1 - k)\ if\ \mu_B(v) < 1 - k$$

from which we get the conclusion. □

Theorem 8. *If the observation contains the premise, i. e.* $\mu_A(u) \le \mu_{A'}(u)\ \forall u \in U$, *then*

$$\mu_{B'}(v) \ge max(\mu_B(v), \frac{1-k}{k}\mu_B(v)(1 - \mu_B(v)))\ if\ \mu_B(v) \le min(0.5, k)$$

$$\mu_{B'}(v) \ge \mu_B(v)\ otherwise.$$

Proof. i1) value on the set $U_1 = \{u \in U / \mu_A(u) \le \mu_B(v)\}$
Because $I_F(\mu_A(u), \mu_B(v)) = 1$ we have

$$\mu_{B'}(v) = \sup_{u \in U_1} min(\mu_{A'}(u), 1) = \sup_{u \in U_1} \mu_{A'}(u) \ge \mu_B(v).$$

i2) value on the set $U_2 = \{u \in U / 0.5 \le \mu_B(v) < \mu_A(u)\} \cup \{u \in U / \mu_A(u) > 1 - \mu_B(v) > 0.5\}$
In this case $I_F(\mu_A(u), \mu_B(v)) = \mu_B(v)$ and

$i2_1$) for $k < \mu_B(v)$ we obtain

$$\mu_{B'}(v) = \sup_{u \in U_2} T_k(\mu_{A'}(u), \mu_B(v)) = \sup_{u \in U_2} min(\mu_{A'}(u), \mu_B(v)) = \mu_B(v)$$

$i2_2$) for $k \ge \mu_B(v)$ we consider two subsets of U_2:

$i2_2^1$) on the subset $U_2^1 = \{u \in U_2 / \mu_{A'}(u) \leq k\}$ we have

$$\mu_{B'}(v) = \sup_{u \in U_2^1} T_k(\mu_{A'}(u), \mu_B(v)) = \sup_{u \in U_2^1} \frac{1-k}{k} \mu_{A'}(u)\mu_B(v)) \leq (1-k)\mu_B(v) < \mu_B(v)$$

$i2_2^2$) on the subset $U_2^2 = \{u \in U_2 / \mu_{A'}(u) > k\}$ we have

$$\mu_{B'}(v) = \sup_{u \in U_2^2} T_k(\mu_{A'}(u), \mu_B(v)) = \sup_{u \in U_2} min(\mu_{A'}(u), \mu_B(v)) = \mu_B(v).$$

i3) value on the set $U_3 = \{u \in U / \mu_B(v) < \mu_A(u) \leq 1 - \mu_B(v)\}$.
In this case $I_F(\mu_A(u), \mu_B(v)) = 1 - \mu_A(u)$ and we analyze the following cases.
i) if $k < \mu_B(v)$ then

$$\mu_{B'}(v) = \sup_{u \in U_3} T_k(\mu_{A'}(u), 1 - \mu_A(u)) = min(\mu_{A'}(u), 1 - \mu_A(u)) < 1 - \mu_B(v).$$

ii) if $k \geq \mu_B(v)$ we consider the following subcases:
ii_1) $\mu_B(v) \leq k \leq 1 - \mu_B(v)$

ii_1^1) on the set $U_3^1 = \{u \in U_3 / \mu_B(v) \leq \mu_A(u) < 1 - k\}$ we have

$$\mu_{B'}(v) = \sup_{u \in U_3^1} min(\mu_{A'}(u), 1 - \mu_A(u)) < 1 - \mu_B(v)$$

ii_1^2) on the set $U_3^2 = \{u \in U_3 / 1 - k \leq \mu_A(u) \leq 1 - \mu_B(v)\}$ we consider two subsets:
• $U_3^{2,1} = \{u \in U_3^2 / \mu_{A'}(u) \leq k\}$ for which we obtain

$$\mu_{B'}(v) = \sup_{u \in U_3^{2,1}} \frac{1-k}{k} \mu_{A'}(u)(1 - \mu_A(u)) \geq \sup_{u \in U_3^{2,1}} \frac{1-k}{k} \mu_A(u)(1 - \mu_A(u))$$
$$\geq max((1-k)^2, \frac{1-k}{k} \mu_B(v)(1 - \mu_B(v))) \geq \frac{1-k}{k} \mu_B(v)(1 - \mu_B(v))$$

• $U_3^{2,2} = \{u \in U_3^2 / \mu_{A'}(u) > k\}$ for which we have

$$\mu_{B'}(v) = \sup_{u \in U_3^{2,2}} min(\mu_{A'}(u), 1 - \mu_A(u)) < 1 - \mu_B(v).$$

ii_2) $k > 1 - \mu_B(v)$ which defines the set

$$U_3^3 = \{u \in U_3 / 1 - k < \mu_B(v)\} = \{u \in U / 1 - k < \mu_B(v) < \mu_A(u) \leq 1 - \mu_B(v)\}$$

• for $\mu_{A'}(u) \leq k$ we obtain

$$\mu_{B'}(v) = \sup_{u \in U_3^3} \frac{1-k}{k} \mu_{A'}(u)(1 - \mu_A(u))$$
$$\geq \sup_{u \in U_3^3} \frac{1-k}{k} \mu_A(u)(1 - \mu_A(u)) \geq \frac{1-k}{k} \mu_B(v)(1 - \mu_B(v))$$

• for $\mu_{A'}(u) > k$ it results

$$\mu_{B'}(v) = \sup_{u \in U_3^3} min(\mu_{A'}(u), 1 - \mu_A(u)) < 1 - \mu_B(v).$$

Finally we obtain the conclusion formulated in the theorem. □

Theorem 9. *If there is a partial overlapping between the sets A and A' then*

$$\mu_{B'}(v) = 1 \; if \; core(A') \cap (U - A_{\mu_B(v)}) \neq \emptyset \; and$$
$$\mu_{B'}(v) \geq \mu_B(v) \; otherwise$$

where A_α denotes the α-cut of A.

Proof. i1) The case $core(A') \cap (U - A_{\mu_B(v)}) \neq \emptyset$.
On the set $U_1 = \{u \in U / \mu_A(u) \leq \mu_B(v)\}$ we have $I_F(\mu_A(u), \mu_B(v)) = 1$ and therefore

$$\mu_{B'}(v) = \sup_{u \in U_1} T_k(\mu_{A'}(u), 1) = 1.$$

i2) The case $core(A') \cap (U - A_{\mu_B(v)}) = \emptyset$.
On the set $U_2 = \{u \in U / \mu_A(u) > \mu_B(v) \geq 0.5\}$ we have $I_F(\mu_A(u), \mu_B(v)) = \mu_B(v)$ and therefore

$$\mu_{B'}(v) = \sup_{u \in U_2} T_k(\mu_{A'}(u), \mu_B(v)) \geq T_k(1, \mu_B(v)) = \mu_B(v).$$

If $\mu_B(v) < 0.5$ we analyze three cases. Let $\tilde{U} = \{u \in U / \mu_A(u) = \mu_B(v)\}$; $card(\tilde{U}) = 2$ if $0 < \mu_B(v) < 1$.
$i2_1$) the case $\tilde{U} \cap supp(A') = \emptyset$ and $core(A') \cap core(A) \neq \emptyset$.
On the set $U_3 = \{u \in U / \mu_A(u) \geq 1 - \mu_B(v) > 0.5\}$ it results

$$\mu_{B'}(v) = \sup_{u \in U_3} T_k(\mu_{A'}(u), \mu_B(v)) \geq T_k(1, \mu_B(v)) = \mu_B(v).$$

$i2_2$) the case $\tilde{U} \cap supp(A') = \emptyset$ and $core(A') \cap core(A) = \emptyset$.
We consider the set $U_4 = \{u \in U / \mu_B(v) < \mu_A(u) \leq 1 - \mu_B(v)\}$; on the set $U_5 = U_3 \cup U_4$ we have

$$\mu_{B'}(v) = \sup_{u \in U_5} T_k(\mu_{A'}(u), I_F(\mu_A(u), \mu_B(v))) \geq \sup_{u \in U_5} T_k(\mu_{A'}(u), \mu_B(v)) \geq T_k(1, \mu_B(v)) = \mu_B(v).$$

$i2_3$) the case $\tilde{U} \cap supp(A') \neq \emptyset$. On the set U_5 we obtain $\mu_{B'}(v) \geq \mu_B(v)$, as in the previous case.
It results that, in the case i2), $\mu_{B'}(v) \geq \mu_B(v)$. The same result is obtained for $\mu_B(v) \in \{0, 1\}$. □

We consider the negation with threshold $k \in (0, 1)$ [31]

$$N_k(x) = \begin{cases} 1 - \frac{1-k}{k}x & if \ \ x \leq k \\ \frac{k}{1-k}(1-x) & if \ \ x \geq k \end{cases}$$

obtained from the standard negation $N(x) = 1 - x$.

Theorem 10. *If the premise and the observation are contradictory, i.e.* $\mu_{A'}(u) = N_k(\mu_A(u))\ \forall u \in U$, *then* $\mu_{B'}(v) = 1\ \forall v \in V$.

Proof. On the set $U_1 = \{u \in U / \mu_A(u) \leq \mu_B(v)\}$ we have

$$\mu_{B'}(v) = \sup_{u \in U_1} T_k(\mu_{A'}(u), 1) = \sup_{u \in U_1} min(\mu_{A'}(u), 1) = \sup_{u \in U_1} \mu_{A'}(u) = \sup_{u \in U_1} N_k(\mu_A(u)) = 1$$

because there is $u_0 \in U_1$ with $\mu_A(u_0) = 0$. □

4 Interpretation and utilization of results

In this section we will compare the results given by the common operators (t-norm product $T_P(x, y) = xy$ and negation $N(x) = 1 - x$) with those obtained by the corresponding operators with threshold and we will indicate some possibility of their utilization in a fuzzy reasoning system. An example of working with these results is also presented. In the case of standard operators T_P and N, according to [24] we have:

Theorem 11. *If the premise contains the observation, i.e* $\mu_{A'}(u) \le \mu_A(u) \quad \forall u \in U$*, then*

$$\mu_{B'}(v) = \mu_B(v) \; if \; \mu_B(v) \ge 0.5 \; or \; (0.25 \le \mu_B(v) < 0.5)$$

$$\mu_{B'}(v) < 0.25 \; if \; \mu_B(v) < 0.25$$

Theorem 12. *If the premise and the observation coincide, i.e.* $\mu_A(u) = \mu_{A'}(u) \; \forall u \in U$*, then*

$$\mu_{B'}(v) = \max(\mu_B(v), 0.25)$$

Theorem 13. *If the observation contains the premise, i.e.* $\mu_A(u) \le \mu_{A'}(u) \; \forall u \in U$*, then*

$$\mu_{B'}(v) \ge \mu_B(v) \; \forall v \in V.$$

Theorem 14. *If there is a partial overlapping between the sets* A *and* A'*, then*

$$\mu_{B'}(v) = 1 \; if \; core(A') \cap (U - A_{\mu_B(v)}) \ne \emptyset \; and$$

$$\mu_{B'}(v) \ge \mu_B(v) \; otherwise$$

where A_α *denotes the* $\alpha-$*cut of* A*.*

Theorem 15. *If the premise and the observation are contradictory, i.e.* $\forall u \in U \; \mu_{A'}(u) = 1 - \mu_A(u)$*, then* $\mu_{B'}(v) = 1 \; \forall v \in V$*.*

If the observation is more precise than the premise of the rule then it gives more information than the premise. However, it does not seem reasonable to think that the Generalized Modus Ponens allows to obtain a conclusion more precise than that of the rule. The result of the inference is valid if $\mu_{B'}(v) = \mu_B(v)$, $\forall v \in V$. Sometimes, the deduction operation allows the reinforcement of the conclusion, as is specified in [28], [19] and [25]:

Rule: *If the tomato is red then the tomato is ripe.*

Observation: *This tomato is very red.*

If we know that the maturity degree increases with respect to color, we can infer "*this tomato is very ripe*". On the other hand, in the example

Rule: *If the melon is ripe then it is sweet*

Observation: *The melon is very ripe*

we do not infer that "*the melon is very sweet*" because it can be so ripe that it can be rotten.

This examples show that if the expert has not supplementary information about the connection between the variation of the premise and the conclusion, he must be satisfied with the conclusion $\mu_{B'}(v) = \mu_B(v)$. The Theorem 6 gives a valid result if we choose $\mu_{B'}(v) = \mu_B(v)$ for $\mu_B(v) < 0.5$. As opposite, the corresponding Theorem 11 from the case of the standard t-norm T_P does not allow to obtain a valid result if $\mu_B(v) < 0.25$.

When the observation and the premise of the rule coincide the convenient behavior of the fuzzy deduction is to obtain an identical conclusion. A different conclusion indicates the appearance of an uncertainty in the conclusion. The both theorems, 7 and 12, give an uncertain conclusion, but we can choose $k > 0.75$ in the Theorem 7 and we obtain a better result, because the uncertainty is smaller in comparison with the result from the Theorem 12.

If the observation contains the premise, because

$$max(\frac{1-k}{k}\mu_B(v)(1-\mu_B(v)), \mu_B(v)) \ge \mu_B(v)$$

it results that Theorem 8 gives a better result that Theorem 13. In this case the inferred conclusion B' is a superset of B; we can choose the first superset.

If there is a partial overlapping between the premise and the observation or the premise and the observation are contradictory then the two t-norms give the same results for the inferred conclusion. The value $\mu_{B'}(v) = 1$ obtained in these cases represents an indeterminate conclusion, all elements $v \in V$ having a possibility equal to 1. In the case of "partial overlapping" we propose a "mediation" between the two possible values:

$$\mu_{B'}(v) = 1 \ and \ \mu_{B'}(v) \geq \mu_B(v);$$

if B_1, B_2, ..., B_k are the supersets of B with

$$\mu_{B_k}(v) \geq \mu_{B_{k-1}}(v) \geq .... \geq \mu_{B_1}(v),$$

we can choose $B' = B_{[\frac{k}{2}]}$, where $[x]$ is the greatest integer which is smaller than or equal to x. The Theorem 10 gives a waited result, that represents one of the basic properties of GMP reasoning.

The results from Theorems 6-10 can be used in a fuzzy inference system as in the following example. A customer is interested to buy a computer. The quality of the computer depends on its price as is specified by the rules:

Rule1: If the price is very low then the quality is below average
Rule2: If the price is very very high then the quality is very good.
Rule3: If the price is middle then the quality is good.

The variable price has values in the following set of linguistic terms

$$L_p = \{very\ very\ low, very\ low, low, midle, high, very\ high, very\ very\ high\}$$

and the variable quality has values in the set

$$L_q = \{poor, below\ average, average, above\ average, good, very\ good\}.$$

We consider the universes of discourse $[0, 2200]$ for price and $[0, 10]$ for quality. The linguistic terms are represented by the following trapezoidal fuzzy numbers:

$very\ very\ low = (0, 100, 0, 100)$
$very\ low = (0, 200, 0, 200)$
$low = (400, 700, 100, 100)$
$middle = (900, 1300, 300, 300)$
$high = (1500, 1700, 200, 200)$
$very\ high = (2000, 2200, 200, 0)$
$very\ very\ high = (2100, 2200, 100, 0)$
$poor = (1, 9, 1, 1)$
$below\ average = (3, 7, 2, 2)$
$average = (3.5, 6.5, 1.5, 1.5)$
$above\ average = (4, 6, 1, 1)$
$good = (4.5, 5.5, 0.5, 0.5)$
$very\ good = (4.75, 5.25, 0.5, 0.5)$.

These fuzzy numbers are depicted in the Figures 1 and 2.

We consider the observations:

Observation1: the price is very very low
Observation2: the price is very high
Observation3: the price is high

The theorems 6-10, used together with the comments from this section, give the following results:

1) the conclusion obtained from Rule1 and Observation1 is "the quality is below average"; this result is obtained with Theorem 6

2) Theorem 8 is applied for Rule2 and Observation2 and gives the conclusion "the quality is good"

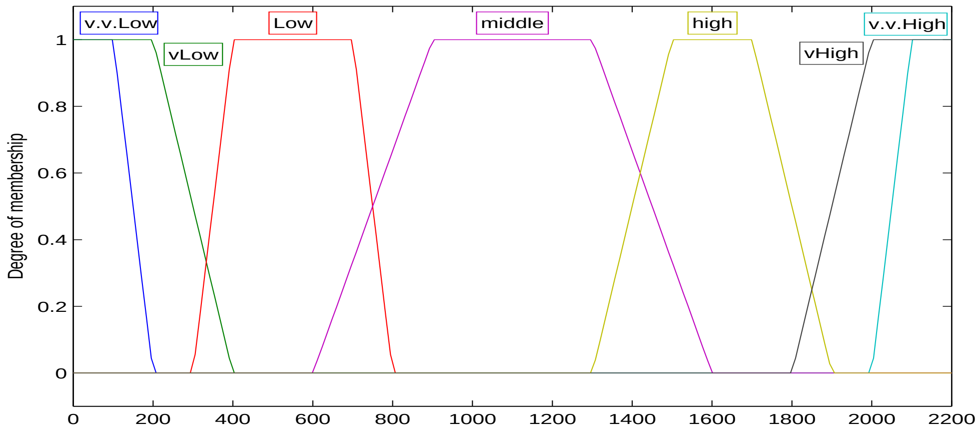

Figure 1: Fuzzy sets for linguistic terms from the list L_p

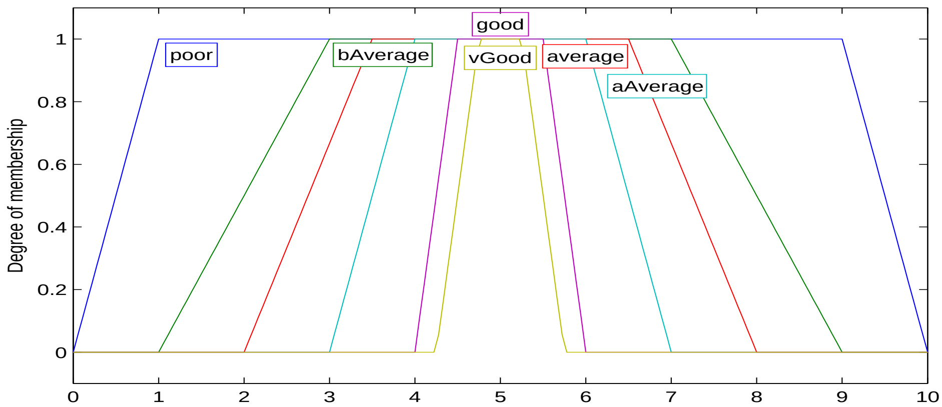

Figure 2: Fuzzy sets for linguistic terms from the list L_q

3) using Theorem 9 for the Rule3 and Observation3 one obtain the conclusion "the quality is average".

As it can be observed from this example, our results allow us to obtain the inferred conclusion by a very simple calculus in comparison with the standard formula used in GMP.

5 Summary and Conclusions

The results obtained in this paper explain how the Generalized Modus Ponens rule works with the Fodor's implication and the t-norm product with threshold. Combining these results with the approximations proposed in the previous section we obtain a fast answer for the value of the conclusion inferred by GMP reasoning. We worked with the t-norm product because it is one of the most used in practical applications. As it results from the previous sections, one obtain better results in the case of t-norm with threshold. In a future paper we will analyze the results given by another t-norms with threshold and another implications.

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